Appearance and disappearance of the Fermi statistics: The BCS-BEC crossover

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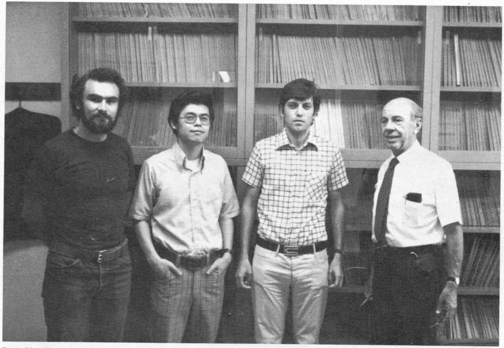
“Centenary of the Fermi Statistics”
Accademia dei Lincei, Rome
February 5-6, 2026

Some personal memories

My personal memories of Enrico Fermi are "at second order" ⟸

I have closely interacted with two Fermi's collaborators:

ⓐ Ugo Fano (theoretician), the last of "Rome Via Panisperna boys" (emigrated to the USA before WW2), was my supervisor and mentor at the University of Chicago (1974-78)



Prof. Ugo Fano with graduate students Costas Theodosiou; post-doc Tu-Nan Chang; and brand new graduate student, Giancarlo Strinati.

Just entering Graduate School at the University of Chicago (Sept. 1974)

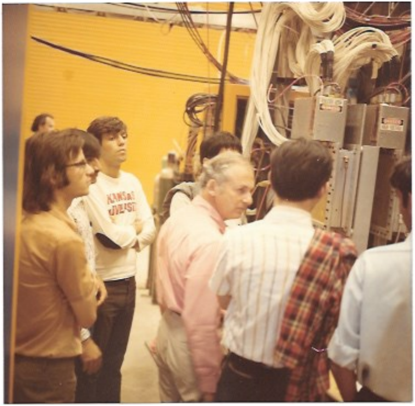
Some personal memories

ⓑ Herbert Anderson (experimentalist), Fermi's first graduate student in the USA (at Columbia University) and then Fermi's close collaborator up to Fermi's last days (1954)

Fermi and Anderson (next to Leó Szilárd, next to Leona Woods) in front of the Eckhart Hall at the University of Chicago

Some personal memories

Anderson supervised my experimental course at the University of Chicago



Visiting Fermilab (Batavia) in July 1975

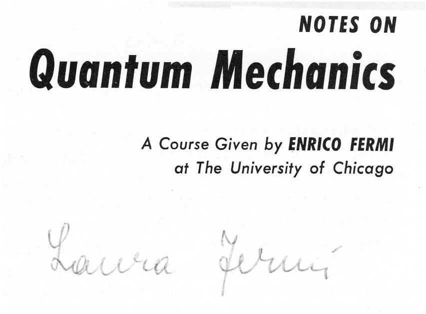
I had the option of becoming a theoretician (with Fano)
or an experimentalist (with Anderson)

$\Longrightarrow$ I ended up being Fano's student and then post-doc

Meeting with Laura Fermi

Through the Fanos, I had also the opportunity of meeting with Enrico's wife Laura, who told me a few personal memories about her life with Enrico $\cdots$

In that occasion, I asked Laura to sign my own copy of Enrico's last Lectures Notes at the University of Chicago (1954):



[I will be back to this booklet shortly]

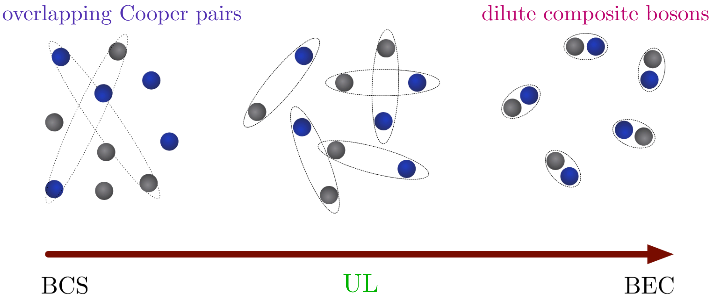
The emergence of the BCS-BEC crossover

Several years later (about 2003), I came across the emerging field of the BCS-BEC crossover with ultra-cold Fermi gases

$\Longrightarrow$ a continuous evolution was experimentally realized

from weakly-bound strongly overlapping Cooper pairs (BCS)

to tightly-bound dilute dimers undergoing Bose-Einstein condensation (BEC)



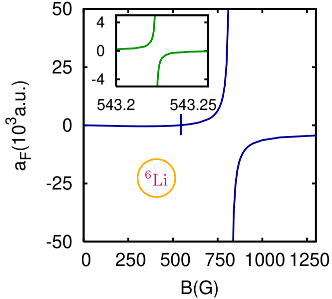
Evolution from BCS to BEC, passing through "unitarity"

[Co-workers in this enterprise: Pierbiagio Pieri, Andrea Perali, $\cdots$]

The Fano-Feshbach resonances

An essential experimental tool $\Longleftrightarrow$ the Fano-Feshbach resonances:

the *scattering length a_F* for the two-fermion problem can be varied at will



from 0^- (free fermions) to 0^+ (strongly bound fermions)

The other relevant length scale = *average inter-particle distance* $\longleftrightarrow$

inverse of Fermi wave vector $k_F = \left(3\pi^2 n\right)^{1/3}$ (n = particle density) $\Longrightarrow$

coupling parameter $(k_F a_F)^{-1}$ ranging from $(k_F\, a_F)^{-1} \lesssim -1$ (BCS regime, $a_F < 0$)

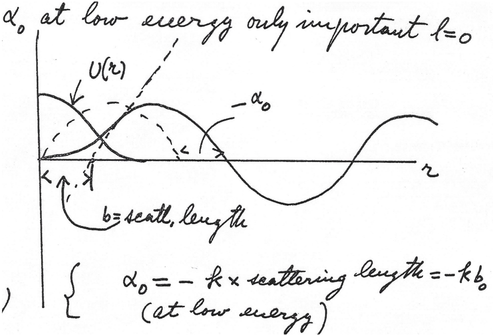
to $(k_F\, a_F)^{-1} \gtrsim +1$ (BEC regime, $a_F > 0$)

across unitarity, $|a_F| \approx \infty$

Original Fermi's drawing for the scattering length

The concept of scattering length was introduced by Fermi in the thirties, to explain some experimental results by E. Segrè on atomic spectroscopy [E. Fermi, *Sopra lo spostamento per pressione delle righe elevate delle serie spettrali*, Nuovo Cimento **11**, 157 (1934)].

Here is Fermi's hand-drawing from his last course at the University of Chicago (1954):



[From E. Fermi, *Notes on Quantum Mechanics* (The University of Chicago Press, Chicago, 1961), page 33-3]

A remark from the original BCS article (1957)

The original BCS article [Phys. Rev. **108**, 1175 (1957)] was rather negative about the connection between superconductivity and BEC.

Here is footnote 18 from this article:

"Our picture differs from that of Schafroth, Butler, and Blatt$^{(*)}$, who suggest that pseudo-molecules of pairs of electrons of opposite spin are formed.

They show if the size of the pseudo-molecules is less than the average distance between them, and if other conditions are fulfilled, the system has properties similar to that of a charged Bose-Einstein gas, including a Meissner effect and a critical temperature of condensation.

Our pairs are not localized in this sense, and our transition is not analogous to a Bose-Einstein condensation."

$^{(*)}$ M. R. Schafroth, S. T. Butler, and J. M. Blatt, *Quasi-chemical equilibrium model to superconductivity*, Helv. Phys. Acta **30**, 93 (1957)

The "underlying" Fermi surface

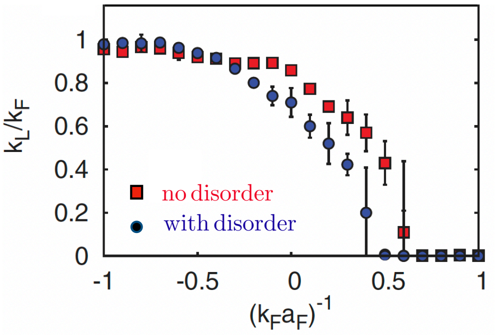
What BCS had in mind to emphasize was that, for the weak-coupling superconductors known at the time,

the *"underlying" Fermi surface* is an essential ingredient of the theory

(although somewhat "blurred" and not sharp like in a normal Fermi liquid).

However, in the BCS-BEC crossover

the underlying Fermi surface "**collapses**" upon approaching the BEC limit:



BCS (mean-field) description of the ground state

Notwithstanding the BCS comment, the BCS (ground state) wave function

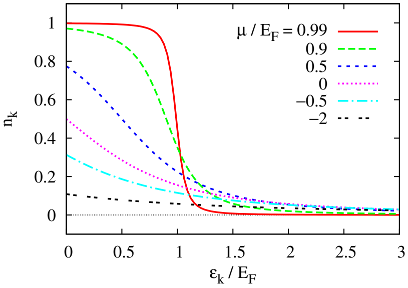
$$|\Phi_{\rm BCS}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right)|0\rangle \propto \exp\left[\sum_{\mathbf{k}} g_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right]|0\rangle$$

contains the BEC limit (at $T = 0$). Here,

$$b_0^\dagger \equiv \sum_{\mathbf{k}} g_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \quad \text{with} \quad [b_0, b_0^\dagger] = \sum_{\mathbf{k}} |g_{\mathbf{k}}|^2 (1 - n_{\mathbf{k}\uparrow} - n_{-\mathbf{k}\downarrow})$$

is **not** a truly bosonic operator. Yet,

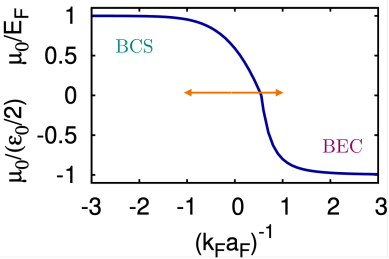
it may happen that $[b_0, b_0^\dagger] \cong 1 \quad \Longleftarrow \quad \langle \Phi_{\rm BCS}|n_{\mathbf{k}\sigma}|\Phi_{\rm BCS}\rangle = n_{\mathbf{k}} \ll 1$ for all **k**



Evolution of the occupation number $n_{\mathbf{k}}$ for different values of the chemical potential μ

The special role played by the chemical potential

As shown in the previous figure, the chemical potential $\longleftrightarrow$ a *driving field* that induces an evolution BCS $\longleftrightarrow$ BEC



Note however that in the BEC limit of the BCS wave function:

- the pair "internal" degrees of freedom are frozen with **zero** center-of-mass momentum
  $\Longrightarrow$
- the *condensate* is accounted for, but the *non-condensate* with **non-zero** center-of-mass momenta is not $\Longrightarrow$
- the need arises to include (beyond-mean-field) *"pairing fluctuations"* $\Longrightarrow$
- the *t*-matrix approach for "dilute" fermions with short-range attraction is a candidate

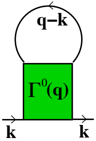
Beyond mean-field → the t-matrix approximation

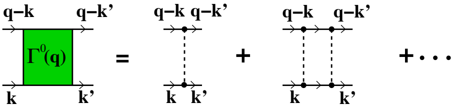
Scattering of two fermions in the presence of the medium through which they propagate

⟹ the *t-matrix approximation* was

- originally formulated by Galitskii (1958) for a repulsive interaction
- later extended by Gorkov & Melik-Barkhudarov (1961) to an attractive interaction to deal with fermionic superfluidity (in the BCS limit only)
- first applied to the BCS-BEC crossover by Nozières & Schmitt-Rink (1985) to recover the correct value of the critical temperature in the BEC limit

Here is the diagrammatic representation of the t-matrix :





The “underlying” Fermi surface $\rightarrow$ the Luttinger wave vector

- The *“Luttinger wave vector”* k_L highlights the **presence** of an **underlying Fermi surface** in the single-particle excitations $\Longrightarrow$

  the last remnant of what would be a Fermi-liquid description of a Fermi gas

- look at the ω-structures of the single-particle spectral function $A(k,\omega)$ for given k

- fit the *dispersions* of the peaks at $\omega > 0$ and $\omega < 0$ with the BCS-like expressions

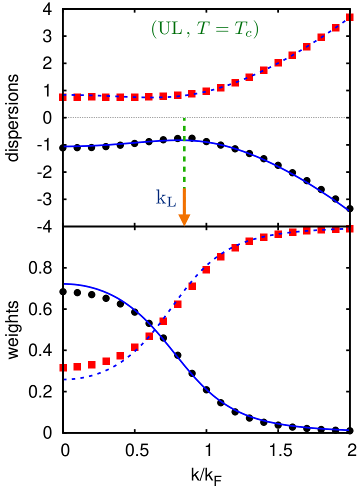
$$\omega_{(\pm)}(k) = \pm \sqrt{\left(\frac{k^2}{2m} - \frac{k^2_{L(\pm)}}{2m}\right)^2 + \Delta^2_{\mathrm{pg}(\pm)}}$$

$\Delta_{\mathrm{pg}(\pm)} \longleftrightarrow$ pseudo-gap energies for the upper $(+)$ and lower $(-)$ branches

$\Longrightarrow$ identify $k_{L(+)}$ for the “up-bending” upper branch

$k_{L(-)}$ for the “down-bending” lower branch $\qquad (k_{L(+)} < k_{L(-)})$

Single-particle spectral function & Co.



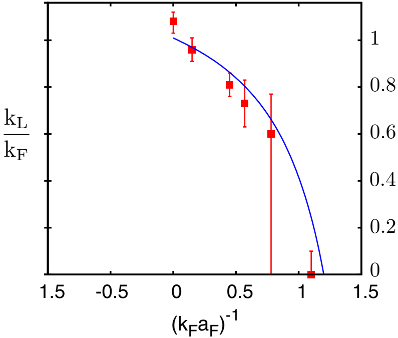
circles (squares) ⟷ numerical calculations

—— (- - - -) lines ⟷ BCS-like fits

An experimental confirmation

ARPES-like experiments with ultra-cold trapped Fermi gases

(work in collaboration with Debbie Jin group[*])



coupling dependence of the Luttinger wave vector k_L at T_c

experiment (squares) theory (solid line)

[*] A. Perali, F. Palestini, P. Pieri, G. C. Strinati, J. T. Stewart, J. P. Gaebler, T. E. Drake, and D. S. Jin, *Evolution of the Normal State of a Strongly Interacting Fermi Gas from a Pseudogap Phase to a Molecular Bose Gas*, Phys. Rev. Lett. **106**, 060402 (2011)

Self-consistency for the t-matrix $\longrightarrow$ to be or not to be !

- In diagrammatic many-body theory, calculating a self-energy up to full self-consistency is a debated issue $\Longrightarrow$ the t-matrix approximation makes no exception !

  [M. Pini, P. Pieri, and G. Calvanese Strinati, *Fermi gas throughout the BCS-BEC crossover: Comparative study of t-matrix approaches with various degrees of self-consistency*, Phys. Rev. B **99**, 094502 (2019)]

- A pragmatic way to settle this issue would be to compare alternative calculations with the available experimental data

- For instance, for the critical temperature T_c *at unitarity* one obtains:

  non-self-consistent t-matrix calculation $\Longrightarrow$ $T_c/T_F \simeq 0.24$

  fully-self-consistent t-matrix calculation $\Longrightarrow$ $T_c/T_F \simeq 0.16$

  (T_F = Fermi temperature)

  to be compared with the experimental value $T_c/T_F \simeq 0.17$

- However, additional diagrammatic contributions may influence the value of T_c !

The "original" Gor'kov & Melik-Barkhudarov contribution

The diagrammatic contribution considered by Gor'kov and Melik-Barkhudarov (GMB)*
(but only in the BCS limit) modifies the BCS value of T_c by a factor of 2.2 $\Longrightarrow$

it is thus appropriate to *"extend"* the GMB contribution to the whole BCS-BEC crossover

$\Longleftarrow$ exploit a main advantage of the diagrammatic theory for being "modular" in nature !

The critical temperature T_c is determined from the normal phase via the Thouless criterion

$$\Longleftrightarrow$$

look for divergences at long wavelength of the static ladder propagator $\Gamma^0(Q = 0)$

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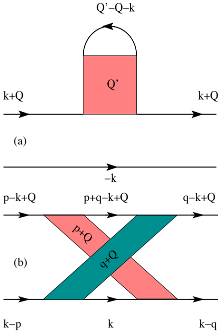
* L. P. Gor'kov and T. M. Melik-Barkhudarov, *Contribution to the theory of superfluidity in an imperfect Fermi gas*, Sov. Phys. JETP **13**, 1018 (1961)

The “extended” Gor’kov & Melik-Barkhudarov contribution

Improve on the Thouless criterion by dressing $\Gamma^0(Q)$ with “bosonic-like” self-energies

$\Longrightarrow$ above T_c the “extended” GMB contribution amounts to inserting the following (bosonic-like) diagrams in the ladder propagator Γ^0



[the upper diagram accounts for some degree of self-consistency in the t-matrix]

Include the full wave-vector and frequency dependence of **all** Γ^0 in these diagrams

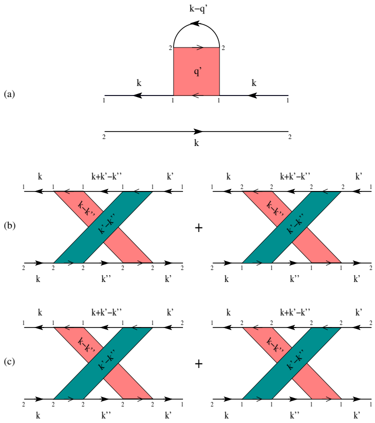
$\Longrightarrow$ the GMB approach is “extended” to the whole BCS-BEC crossover

A novel approach to the gap equation

Below T_c, the “extended” GMB contribution determines the gap equation directly in

the *two-particle channel* $\Longleftrightarrow$ Hugenholtz-Pines condition for fermion pairs

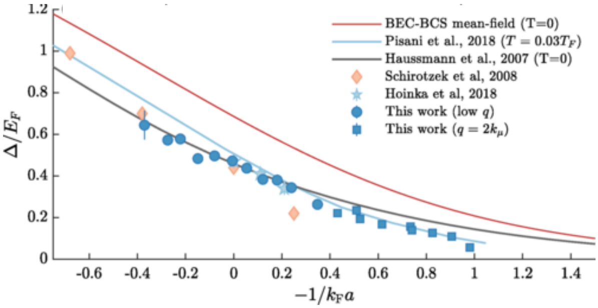
(analogy with point-like bosons)



[numbers attached to vertices $\longleftrightarrow$ Nambu indices]

Comparison with experiments - pairing gap

Low-temperature pairing gap from BCS to BEC

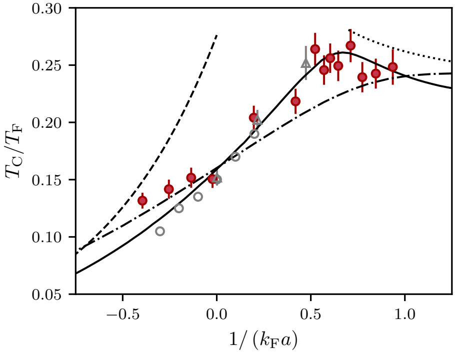


Measurements of the low-temperature pairing gap Δ across the BCS-BEC crossover for a balanced spin mixture of an ultra-cold gas of $^6\mathrm{Li}$ atoms (note the sign change for the inter-particle coupling). Comparison with three theoretical results is also reported.

From Fig. 3 of H. Biss, L. Sobirey, N. Luick, M. Bohlen, J. J. Kinnunen, G. M. Bruun, T. Lompe, and H. Moritz, *Excitation spectrum and superfluid gap of an ultracold Fermi gas*, Phys. Rev. Lett. **128**, 100401 (2022).

Comparison with experiments - critical temperature

Critical temperature from BCS to BEC



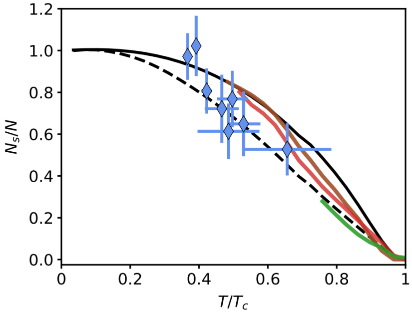
Measurements of the critical temperature T_c for an ultra-cold Fermi gas spanning the BEC-BCS crossover are compared with the results of theoretical calculations:

- fully-self-consistent *t*-matrix approach (dashed-dotted line $- \cdot - \cdot - \cdot$)
- extended GMB approach (full line ———)

From Fig. 2 of M. Link, K. Gao, A. Kell, M. Breyer, D. Eberz, B. Rauf, and M. Köhl, *Machine learning the phase diagram of a strongly interacting Fermi gas*, Phys. Rev. Lett. **130**, 203401 (2023).

Comparison with experiments - superfluid fraction

Superfluid fraction vs temperature at unitarity:



Measurements of the temperature dependence of the superfluid density for a Fermi gas at unitarity are compared with the results of theoretical calculations:

- non-self-consistent *t*-matrix approach (dashed line – – – –)
- extended GMB approach (black full line ——)

From Fig. 4 of M. Frómeta Fernández, D. Hernández-Rajkov, G. Del Pace, N. Grani, M. Inguscio, F. Scazza, S. Stringari, and G. Roati, *Angular momentum of rotating fermionic superfluids by Sagnac phonon interferometry*, arXiv:2511.02664v2.

n.b. In all cases, the extended GMB calculations are with no adjustable parameter

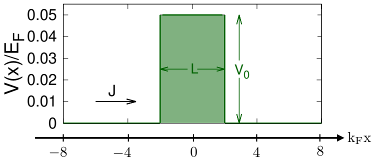
Extension to inhomogeneous situations

All considerations thus far were for homogeneous systems.

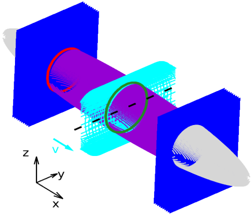
In practice, inhomogeneous situations are abundant and important.

Typically, a barrier is required for the Josephson effect with ultra-cold Fermi atoms.

A simplified situation:



A realistic situation:



The Local Phase Density Approximation (LPDA)

Fermionic superfluids in inhomogeneous environments $\Longrightarrow$ BdG equations are often used

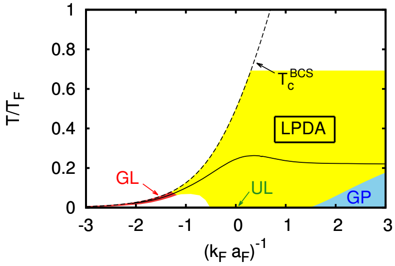
However, their solution may become prohibitive $\Longleftarrow$ exceeding computation time and memory space $\Longleftarrow$ first calculate a lot of spatial details and then average over them !

To overcome these difficulties $\Longrightarrow$ apply a coarse graining on the BdG equations throughout BCS-BEC crossover $\Longrightarrow$

obtain a (non-linear) differential equation for $\Delta(\mathbf{r})$ $\Longleftrightarrow$ generalize

- the Ginzburg-Landau (GL) equation for strongly overlapping Cooper pairs
- the Gross-Pitaevskii (GP) equation for dilute composite bosons

This (LPDA) equation holds over a wide region of the temperature-coupling phase diagram:

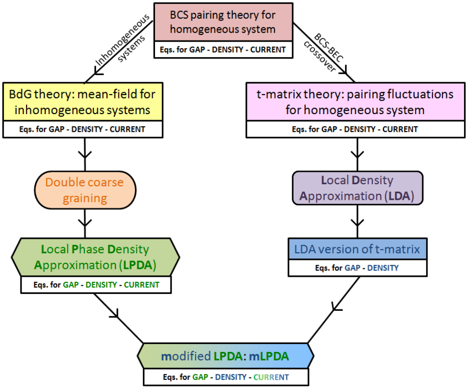


A further step: beyond the LPDA $\rightarrow$ the mLPDA

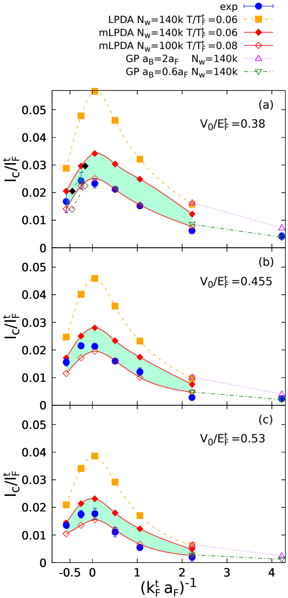
One would also like to include pairing fluctuations within the LPDA approach $\Longrightarrow$

- keep the formal structure of the LPDA equation
- modify the expressions for the local particle density and current where the effects of pairing fluctuations are included

$\Longrightarrow$ obtain the mLPDA approach (m $\leftrightarrow$ modified)



Comparison with experiments - the Josephson effect



Critical current vs trap coupling for three barriers of same width and different heights.

Experimental data (•) are compared with theoretical results of LPDA (■) and mLPDA (♦ and ◊) approaches.

Theoretical results are obtained for different temperatures and atoms number N_w within experimental ranges

$\Longrightarrow$ shaded areas spanned by numerical calculations.

In (a), ♦ and ◊ correspond to a simplified version of the extended GMB approach.

[Experiment: W. J. Kwon *et al.*, Science **369**, 84 (2020)]

Thank you for your attention !