Evolution Equations in Climate and Environmental Sciences

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ABSTRACTS

Solar Variability as Climate Forcing: Bolometric and Spectral Perspectives Across Time Scales

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Solar radiation is the dominant term in Earth's energy budget and a key external forcing in the climate system, yet its output is not constant across time. This talk offers an overview of solar irradiance variability, both bolometric (Total Solar Irradiance, TSI) and spectral (SSI), across the full hierarchy of physical timescales on which it manifests itself: the secular brightening of the Sun as a main-sequence star over billions of years; the transition, common to solar-like stars, from a spot-dominated to a faculae-dominated magnetic regime; the 11-year activity cycle together with its associated Gleissberg- and Suess–de Vries-scale modulations; and the impulsive, short-lived variations associated with flares. Since direct radiometric measurements only became available with space-based instruments in the late 1970s, extending the record backward in time relies on proxies, such as historical sunspot observations, chromospheric solar image archives, and cosmogenic isotopes (^{14}C, ^{10}Be) preserved in tree rings and ice cores. We show how these proxies are combined within semi-empirical models to reconstruct TSI and SSI over the Holocene, and in particular over the last millennium, including grand minima such as the Maunder and Spörer minima, comparing recent independent reconstructions and discussing their associated uncertainties. We conclude by considering the role of solar forcing within the paleoclimate record: although detectable, it appears secondary to volcanic and anthropogenic greenhouse gas forcing over recent centuries, and current TSI reconstructions do not support solar variability as a driver of the warming trend observed since the mid-twentieth century.

Continuous data assimilation with multiplicative observation noise

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Data assimilation aims to reconstruct the evolving state of a dynamical system by combining a mathematical model with partial and noisy observations. In continuous data assimilation, this can be done through a nudging mechanism: an approximate solution is continuously corrected using the discrepancy between its observed components and those of the true system.

We study this problem for dissipative evolution equations in a Gelfand triple. The reference solution is observed only through finitely many modes, and the observations are affected by multiplicative stochastic noise. We give conditions under which the nudged approximation nevertheless synchronises with the true solution. In particular, we derive quantitative mean-square error estimates and prove almost sure synchronisation when the observation noise becomes sufficiently small along the reference trajectory.

Our abstract framework covers both weak and strong formulations of several PDEs. This is joint work with J. Bröcker (Uni. of Reading), F. Palma (Uni. of Campania), M. Hieber, and T. Zöchling (TU Darmstadt).

Reconstruction of degeneracy in parabolic equations

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We address an inverse problem involving the reconstruction of a degeneracy in the diffusion coefficient of a one-dimensional parabolic equation, using measurements of the normal derivative on one side of the boundary. Our study focuses on the sensitivity of this inverse problem to the initial data. We establish sufficient conditions on the initial data to ensure uniqueness and stability with a single-point measurement, and provide examples of both positive and negative results. Furthermore, we present more general uniqueness results for the simultaneous identification of both the degeneracy and the initial data using boundary measurements distributed over time. We also present some generalizations to a system of equations and to two-dimensional radial cases. These theoretical conclusions are supported by numerical evidence. The method of proof is based on the representation of the solution by means of Bessel functions of the first kind. This is joint work with P. Cannarsa, M. Yamamoto, and V. Danesi.

Variational Analysis of Transition Climates in 1-D Degenerate and 2-D Spherical Energy Balance Models

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In this talk, we analyze the structure of stationary solutions for a class of Budyko-Sellers Energy Balance Models with spatially heterogeneous insolation. We provide a rigorous variational framework for the problem, considering both a one-dimensional setting with degenerate diffusion at the poles and a two-dimensional formulation governed by the Laplace-Beltrami operator on the sphere. By establishing Palais-Smale compactness in weighted Sobolev spaces in the 1-D case, we prove the existence of three distinct climate states: two stable equilibria (warm and fully glaciated) and an unstable transition climate characterized as a mountain-pass critical point. Furthermore, we are able to establish the persistence of this unstable structure under small heterogeneous perturbations of the solar forcing. This is joint work with Gianmarco Del Sarto (Darmstadt).

On the null controllability of non-autonomous degenerate PDEs with a climatological application

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Inspired by a Budyko-Seller model, we consider non-autonomous degenerate parabolic problems. Using Kato's Theorem, we first prove the well-posedness of such problems. Then, obtaining new Carleman estimates for the non-homogeneous adjoint problems, we deduce null-controllability for the original ones. Some linear and semilinear extensions are also addressed, as well as open problem and work in progress.

Inverse Problems for Complex Media: Stability, Interfaces and Unique Continuation

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Inverse problems for heterogeneous media often require the recovery of internal information from partial, indirect or boundary measurements. This leads to fundamental analytical questions concerning stability, unique continuation, propagation of smallness and the effect of interfaces or discontinuous coefficients.

In this talk, I will present an overview of some results in this direction, mainly for elliptic and transmission problems. Particular attention will be devoted to quantitative estimates, such as Carleman inequalities, doubling inequalities and Lipschitz stability results, and to their use in the identification of inclusions, interfaces and material parameters.

Although the focus will be on elliptic problems, the methods and questions discussed are naturally connected with broader classes of models for complex media. In particular, they arise in the analysis of stationary regimes, frequency-domain formulations and inverse problems associated with time-dependent phenomena, including models motivated by geophysics and environmental sciences.

Viability and Invariance of Sets under Transport Control Systems in Wasserstein Spaces

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Dynamical models under state constraints arise naturally in sustainable development, population dynamics, robotics, etc. A subset Q is called viable under a dynamical system if for every initial condition in Q there exists a trajectory of the system staying in Q forever. Q is called invariant under a dynamical system if every such trajectory stays in Q forever.

This talk is devoted to necessary and sufficient conditions for viability and invariance of subsets of Borel probability measures under a controlled transport equation on the Wasserstein space.

Multi-Layer Stochastic Energy Balance Models and Uniqueness of the Stationary Measure

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Energy Balance Models (EBMs) focus on the radiation balance of the earth systems. The so-called One-dimensional EBMs simplify the climate system by considering only zonal averages and describe their variation with latitude. Multi-layer models allow for variation in heights by distinguishing between surface and atmospheric temperatures, allowing for a rudimentary representation of the greenhouse effect. The mathematical analysis of the deterministic version of this model has been presented in by C. Urbani [1] in her talk.

In this talk, we investigate the long-time behaviour of this system, see [2]. We consider the above-mentioned two-layer EBM with additive noise acting on both layers and prove the existence and uniqueness of a stationary measure. In fact, we establish exponential mixing by applying a generalized Harris theorem following the lines of Butkovsky, Hairer, Kulik, Mattingly, and Scheutzow.

Finally, I will introduce a coupled ocean-atmosphere model that combines multi-layer EBMs with quasi-geostrophic atmospheric dynamics [3].

References

[1] P. Cannarsa, V. Lucarini, P. Martinez, C. Urbani, J. Vancostenoble, Comparison Principles and Long-Time Behaviour for a Diffusive Energy Balance Model with Vertical Resolution, (2026) arXiv:2604.20339

[2] J. Broeker, P. Cannarsa, G. Carigi, T. Kuna, C. Urbani, A Stochastic Two-Layer Energy Balance Model: well-posedness and exponential ergodicity, (2026) preprint

[3] F. Fornasaro, G. Carigi, T. Kuna, Well-posedness and long time dynamics for a quasi-geostrophic ocean-atmosphere model with radiation balance (2026) arXiv:2512.19556

Recent results for vector-valued Schrödinger operators

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In this talk I will present a series of results on systems of Schrödinger equations, obtained in collaboration with D. Addona (Parma), V. Leone (Salerno), E.M. Ouhabaz (Bordeaux) and A. Rhandi (Salerno).

Linear Response and Optimal Fingerprinting for Nonautonomous Systems

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We provide a link between response theory, pullback measures, and optimal fingerprinting method that paves the way for a) predicting the impact of acting forcings on time-dependent systems and b) attributing observed anomalies to acting forcings when the reference state is not time-independent. We derive formulas for linear response theory for time-dependent Markov chains and diffusion processes. We discuss existence, uniqueness, and differentiability of the equivariant measure under general (not necessarily slow or periodic) perturbations of the transition kernels. Our results allow for extending the theory of optimal fingerprinting for detection and attribution of climate change (or change in any complex system) when the background state is time-dependent and when the optimal solution is sought for multiple time slices at the same time. We provide numerical support for the findings by applying our theory to a modified version of the Ghil-Sellers energy balance model. We verify the precision of response theory - even in a coarse-grained setting - in predicting the impact of increasing CO2 concentration on the temperature field. Additionally, we show that the optimal fingerprinting method developed here is capable to attribute the climate change signal to multiple acting forcings across a vast time horizon.

Domain characterization for Schrödinger operators with sub-quadratic singularity

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We characterize the domain of the Schrödinger operators $S = -\Delta + c|x| - \alpha$ in $L^p(R^N)$, with $0 < \alpha < 2$ and $c \in R$. When $\alpha p < N$, the domain characterization is essentially known and can be proved using different tools, for instance kernel estimates and potentials in the Kato class or in the reverse Hölder class. However, the other cases seem not to be known, so far. In this paper, we give the explicit description of the domain of S for all range of parameters p, α and c.

Improving a Spectral Inequality by Payne

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A celebrated inequality by Payne relates the first eigenvalue of the Dirichlet Laplacian to the first eigenvalue of the buckling problem. Motivated by the goal of establishing a quantitative version of this inequality, we show that Payne's original estimate—which is not sharp—can in fact be improved. Our result provides a refined spectral bound and opens the way to further investigations into quantitative enhancements of classical inequalities in spectral theory.

Perturbation theorem for the essential spectral radius with application to a spatial tumor invasion with cell age

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In this talk, we investigate a spatial diffusion model for tumor invasion in an age-structured population, where the oxygen supply is limited for both proliferating and quiescent tumor cells. We further analyze the model's asynchronous exponential growth behavior.

We first study the essential growth bound of semigroups under boundary perturbations. By combining the variation-of-constants formula from regular system theory with fundamental tools from functional analysis, we develop an operator approximation scheme in which compact operators converge in the operator norm to the convolution operator appearing in the variation-of-constants formula. This approach extends and improves existing results obtained via extrapolation methods by eliminating the need for the norm continuity assumption and by weakening the compactness requirements.

Evolving Measures Corresponding to Nonlinear Semigroups in Forward-Backward Stochastic Dynamics

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This seminar explores a class of infinite-dimensional stochastic partial differential equations driven by Gaussian noise over a bounded domain, characterised by a highly coupled forward-backward structure where the diffusion coefficient depends nonlinearly on the conditional expectation of the terminal state. We first establish the existence and uniqueness of mild, bounded and then we prove that as the initial time is sent to negative infinity, these finite-horizon solutions converge to a unique and stable limit process on the infinite horizon. This asymptotic generates a class of measures and reveals a novel link between the evolution of such measures and nonlinear semigroups.

Analysis of a Two-Layer Diffusive Energy Balance Model

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Energy Balance Models (EBMs) provide a simple yet effective framework for describing the Earth's climate. Introduced by Budyko and Sellers in the late 1960s, classical EBMs describe the evolution of the zonally averaged surface temperature through a one-dimensional diffusive equation.

A natural extension of these models is to include vertical resolution by accounting for the energy exchange between the Earth's surface and the atmosphere. In this talk, I will present the mathematical analysis of a two-layer diffusive EBM. I will first briefly review the main results obtained for the spatially homogeneous model, including well-posedness, positivity, and long-time behaviour [1].

I will then focus on the spatially dependent system [2]. The main ingredient is the establishment of new maximum and comparison principles for coupled degenerate cooperative parabolic systems. These results allow us to prove global well-posedness, preservation of positivity, and the existence of distinguished equilibria. Finally, I will discuss the asymptotic dynamics of the model by proving the existence of a global attractor.

References

[1] P. Cannarsa, V. Lucarini, P. Martinez, C. Urbani, J. Vancostenoble, Analysis of a Two-Layer Energy Balance Model: Long-Time Behaviour and Greenhouse Effect, Chaos, 33 (2023), 113111

[2] P. Cannarsa, V. Lucarini, P. Martinez, C. Urbani, J. Vancostenoble, Comparison Principles and Long-Time Behaviour for a Diffusive Energy Balance Model with Vertical Resolution, (2026) submitted

Characteristic space- and time-scales of atmospheric and climate processes and implications for mathematical and numerical modelling

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Processes governing the evolution of the Earth's atmosphere and climate span a vast range of time and spatial scales, from geological timescales that shape the planet's long-term climate states, through seasonal and daily cycles driven by astronomical forcing, to the rapid changes associated with phenomena such as turbulent fluctuations and droplet growth in clouds. These processes are deeply interconnected, as reflected in the fact that they are often described by the same governing equations, albeit operating at very different scales. In geophysical fluid dynamics, for example, the Navier–Stokes equations can, in principle, describe fluid motions ranging from planetary-scale circulations to the flow around a falling snowflake. Such equations typically contain many terms, each representing a distinct physical process contributing to the dynamics. Their relative importance, however, can vary dramatically across scales. As a result, the "effective" governing equations—those retaining only the dominant processes—may take substantially different forms depending on the regime under consideration. This wide range of scales poses major challenges when these processes must be represented simultaneously, as in numerical weather prediction and climate modelling. This talk will provide an overview of the scales involved in atmospheric and climate dynamics and discuss the mathematical and numerical challenges associated with their modelling. It will also present some of the methods and techniques developed to address these complexities and to incorporate the relevant physical processes into reliable and realistic predictions.